Student Workbook

Converting Products and Sums

Name: ______________________ Date: ______________ Class: ________

ESSENTIAL QUESTION

When is it useful to convert a product of trig functions into a sum, or a sum into a product — and how are these conversions derived?

Lesson Overview

The product-to-sum and sum-to-product formulas are derived by adding and subtracting pairs of sum/difference formulas. They are less commonly memorized than the core identities, but they are essential for specific calculus integration problems and for analyzing wave phenomena in physics — beats, the audible 'wah-wah' effect when two close frequencies play together.

$$\sin\alpha \cos\beta = \tfrac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)]$$

To Sum	$\cos\alpha \sin\beta=\frac{1}{2}[\sin(\alpha+\beta)-\sin(\alpha-\beta)]$; $\cos\alpha \cos\beta=\frac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)]$; $\sin\alpha \sin\beta=\frac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)]$
To Product	$\sin\alpha+\sin\beta=2\sin((\alpha+\beta)/2)\cos((\alpha-\beta)/2)$; $\cos\alpha-\cos\beta=-2\sin((\alpha+\beta)/2)\sin((\alpha-\beta)/2)$
Derivation	Add/subtract the sum & difference formulas for sin or cos, then divide by 2.

Worked Examples

EXAMPLE 1

Write sin(5x)cos(3x) as a sum

- Use $\sin\alpha \cos\beta = \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)]$ with $\alpha=5x$, $\beta=3x$
- $\sin(5x)\cos(3x) = \frac{1}{2}[\sin(5x+3x) + \sin(5x-3x)] = \frac{1}{2}[\sin(8x) + \sin(2x)]$

Answer: $\frac{1}{2}[\sin(8x) + \sin(2x)]$

EXAMPLE 2

Write cos(4x)cos(2x) as a sum

- Use $\cos\alpha \cos\beta = \frac{1}{2}[\cos(\alpha-\beta) + \cos(\alpha+\beta)]$ with $\alpha=4x$, $\beta=2x$
- $\cos(4x)\cos(2x) = \frac{1}{2}[\cos(4x-2x) + \cos(4x+2x)] = \frac{1}{2}[\cos(2x) + \cos(6x)]$

Answer: $\frac{1}{2}[\cos(2x) + \cos(6x)]$

EXAMPLE 3

Write sin(7x) + sin(3x) as a product

- Use $\sin\alpha + \sin\beta = 2\sin((\alpha+\beta)/2)\cos((\alpha-\beta)/2)$ with $\alpha=7x$, $\beta=3x$
- $(\alpha+\beta)/2 = 5x$, $(\alpha-\beta)/2 = 2x$

Answer: $2\sin(5x)\cos(2x)$

EXAMPLE 4

Write $\cos(5x) - \cos(3x)$ as a product

- Use $\cos\alpha - \cos\beta = -2\sin((\alpha+\beta)/2)\sin((\alpha-\beta)/2)$ with $\alpha=5x$, $\beta=3x$
- $(\alpha+\beta)/2 = 4x$, $(\alpha-\beta)/2 = x$

Answer: $-2\sin(4x)\sin(x)$

EXAMPLE 5

Verify: $(\sin(3x) + \sin x)/(\cos(3x) + \cos x) = \tan(2x)$

- Apply sum-to-product to numerator: $\sin(3x) + \sin x = 2\sin(2x)\cos(x)$
- Apply sum-to-product to denominator: $\cos(3x) + \cos x = 2\cos(2x)\cos(x)$
- Divide: $2\sin(2x)\cos(x) / (2\cos(2x)\cos(x)) = \sin(2x)/\cos(2x) = \tan(2x)$ ✓

Answer: Identity verified: $(\sin 3x + \sin x)/(\cos 3x + \cos x) = \tan(2x)$

Guided Practice

Work through each problem below. Use the hint if you get stuck — write your final answer on the last line.

1 Write $\sin(3x)\sin(x)$ as a sum/difference.

Hint: Use $\sin\alpha \sin\beta = ½[\cos(\alpha-\beta) - \cos(\alpha+\beta)]$.

2 Write $\cos(9x) - \cos(5x)$ as a product.

Hint: Use $\cos\alpha - \cos\beta = -2\sin((\alpha+\beta)/2)\sin((\alpha-\beta)/2)$.

3 Write $\sin(6x) - \sin(2x)$ as a product.

Hint: Use $\sin\alpha - \sin\beta = 2\cos((\alpha+\beta)/2)\sin((\alpha-\beta)/2)$.

4 Evaluate: $\sin(75°) + \sin(15°)$ using sum-to-product.

Hint: Apply $\sin\alpha + \sin\beta = 2\sin((\alpha+\beta)/2)\cos((\alpha-\beta)/2)$. The result should simplify to a recognizable exact value.

5 Verify: $(\cos(3x) - \cos x)/(\sin x - \sin(3x)) = \tan(2x)$

Hint: Apply sum-to-product to both numerator and denominator, then simplify.

Key Vocabulary

Product-to-Sum Formula	A formula that rewrites a product of two trig functions as a sum or difference — derived by adding/subtracting sum & difference identities.
Sum-to-Product Formula	A formula that rewrites a sum or difference of two trig functions as a product — the reverse of product-to-sum.
Beat Frequency	The pulsing amplitude heard when two close-frequency waves interfere, modeled by the sum-to-product formula for sinα+sinβ.

Check Your Understanding

Circle the letter of the correct answer for each question.

1 **sinα cosβ equals:** Multiple Choice

- **A** ½[sin(α+β) − sin(α−β)]
- **B** ½[sin(α+β) + sin(α−β)]
- **C** ½[cos(α−β) + cos(α+β)]
- **D** ½[cos(α−β) − cos(α+β)]

2 **sin(5x) + sin(3x) written as a product is:** Multiple Choice

- **A** 2sin(4x)cos(x)
- **B** 2cos(4x)sin(x)
- **C** 2sin(4x)sin(x)
- **D** 2cos(4x)cos(x)

3 **cosα cosβ written as a sum is:** Multiple Choice

- **A** ½[cos(α+β) − cos(α−β)]
- **B** ½[cos(α−β) − cos(α+β)]
- **C** ½[cos(α−β) + cos(α+β)]
- **D** ½[sin(α+β) + sin(α−β)]

4 **cos(7x) − cos(3x) written as a product is:** Multiple Choice

- **A** 2sin(5x)sin(2x)
- **B** −2sin(5x)sin(2x)
- **C** 2cos(5x)cos(2x)
- **D** −2cos(5x)sin(2x)

5 **Product-to-sum formulas are most useful for:** Multiple Choice

- **A** Finding exact values at standard angles
- **B** Integrating products of trig functions in calculus
- **C** Solving trig equations
- **D** Graphing trig functions

Common Mistakes to Avoid

✗ Confusing the product-to-sum formula for $\sin\alpha \sin\beta$ with the one for $\cos\alpha \cos\beta$.

✓ $\sin\alpha \sin\beta = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)]$ (minus). $\cos\alpha \cos\beta = \frac{1}{2}[\cos(\alpha-\beta) + \cos(\alpha+\beta)]$ (plus).

✗ Forgetting the negative sign in $\cos\alpha - \cos\beta = -2\sin(\ldots)\sin(\ldots)$.

✓ This is the only one of the four sum-to-product formulas with a leading negative sign.

✗ Mixing up which function goes in which position in the sum-to-product formulas.

✓ For $\sin\alpha+\sin\beta$: the sum average goes in sine, the half-difference in cosine. For $\cos\alpha+\cos\beta$: both go in cosine.

Math Tips

★ You do not need to memorize all eight formulas. Learn the product-to-sum formula for $\sin\alpha \cos\beta$ and the sum-to-product formula for $\sin\alpha+\sin\beta$ — the others follow the same pattern.

★ The sum-to-product formulas are the reverse of the product-to-sum formulas — derive one set from the other by reading backwards.

★ In physics, $\sin\alpha+\sin\beta=2\sin((\alpha+\beta)/2)\cos((\alpha-\beta)/2)$ models wave superposition: the sine factor is the carrier wave, the cosine factor is the beat envelope.

★ When verifying an identity with a sum or difference of trig functions, try sum-to-product on both numerator and denominator — it often produces dramatic simplification.